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Stable fixed points in models with many coupling constants

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Abstract. Renormalisation group studies in $d = 4 - \varepsilon$ dimensions have thus far indicated that Landau–Ginzburg–Wilson (LGW) models with large numbers of fourth-order invariants do not possess a stable fixed point for small ε. This suggests that the existence of a stable fixed point is simply related to the number of fourth-order invariants. In this paper we show that no such simple relationship exists by constructing LGW models with both arbitrarily large numbers of invariants and a stable fixed point.

Symmetry changes at second-order phase transitions have been a subject of considerable interest. In these transitions the symmetry group, G, of the ordered phase is a subgroup of the symmetry group, G_0, of the disordered phase. The transition is described by an order parameter ψ which determines G; ψ is zero in the disordered phase and non-zero in the ordered phase. According to the theory of Landau and Lifshitz (Landau and Lifshitz 1968, Lifshitz 1942, see also Goshen *et al* 1974) second-order transitions are possible only if the following three conditions are met.

(i) The order parameter ψ transforms as a basis of a *single* irreducible representation, R, of G_0.

(ii) The symmetric part of the representation R^3, denoted $[R^3]$, should not contain the unit representation.

(iii) If the antisymmetric part of R^2, denoted $\{R^2\}$, has a representation in common with the vector representation V, the wavevector q associated with R is *not* determined by symmetry. In this case one expects q to vary continuously in the ordered phase.

Experimental results and model calculations largely confirm the validity of these rules for $d \geq 3$ dimensional systems, where the effects of fluctuations neglected by Landau theory is relatively weak‡. However, in $d = 2$ dimensions, fluctuations are sufficiently strong that violations of the rules are expected even theoretically. For example, the three- and four-state Potts models (Baxter 1973) and the melting transition (Nelson and Halperin 1979) violate the second rule when $d = 2$.

A fourth rule, based on renormalisation group (RG) analysis has been proposed (Halperin *et al* 1974, Mukamel *et al* 1976, Bak *et al* 1976). It states that the absence of

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‡ There exist very few three-dimensional systems in which at least one of the rules appears to be violated experimentally. In 2H-$TaSe_2$, the transition to a CDW state appears experimentally to be second order (Moncton *et al* 1977) despite the existence of a cubic term in the Landau–Ginzburg–Wilson (LGW) Hamiltonian in violation of the second rule (Bak and Mukamel 1979). The transition in NbO_2 is associated with a wavevector, $\boldsymbol{q} = [\frac{1}{4}, \frac{1}{4}, \frac{1}{2}]$, which experimentally does not seem to vary in the ordered phase, thereby violating the third rule (Pynn and Axe 1976, Mukamel 1975). It is, however, possible that the transitions in these systems are weakly first order, and that the $\boldsymbol{q}$ vector in the second example varies slowly with temperature, as the rules would predict.

a stable fixed point in an ε expansion about the upper critical dimensionality for a given phase transition implies that the transition is first order (see Natterman 1976, Rudnick 1978, Iacobson and Amit 1980). This rule provides an explanation for many experimentally observed first-order transitions† (Mukamel 1975, Mukamel and Krinsky 1976, Mukamel *et al* 1976, Mukamel and Wallace 1979, Bak *et al* 1976, Halperin *et al* 1974, Allesandrini *et al* 1976, Brazovskii and Dzyaloshinskii 1975, Mrozinska *et al* 1979, Shnidman and Mukamel 1980). In applying this rule one studies the LGW model associated with the transition within ε expansion (see e.g. Wilson and Kogut 1974). The Hamiltonian for this model typically takes the form

$$H=\int \mathrm{d}^d x \mathscr{H}$$
$$\mathscr{H}=\tfrac{1}{2}\sum_{i=1}^{n}(\nabla\psi_i)^2+\tfrac{1}{2}r\sum_{i=1}^{n}\psi_i^2+\sum_{l=0}^{L-1}u_l f_l(\psi_i) \tag{1}$$

where $\psi_1, \ldots, \psi_n$ are the n components of the order parameter and the $f_l(\psi_i)$ are fourth-order invariants, L in number, of the group G_0. (Note that since the ψ_i form a basis of an irreducible representation of G_0, the quartic invariants, f_l, satisfy the trace condition of Brézin *et al* (1974). The isotropy of the quadratic terms of (1) is therefore preserved under renormalisation group iteration.) The ε expansion can be a lengthy procedure if L is appreciable. However, since the irreducible representation R of the symmetry group G_0 determines the LGW Hamiltonian, it also determines the existence of a stable fixed point. It has therefore been suggested (Bak *et al* 1976) that the fourth rule can be formulated directly in terms of the properties of R (or, in fact, of its character table). Such a formulation would place the fourth rule on the same footing as the first three: no RG calculation would be required. This goal has, thus far, proven unattainable.

Experience with ε-expansion calculations indicates that LGW models with sufficiently large numbers of fourth-order invariants do not possess a stable fixed point to leading order in ε. Indeed, to our knowledge no n-component model with $L>3$ *and* a stable fixed point has ever been found for $n\neq 0$‡. It is therefore tempting to conjecture that recursion relations involving more than three (or perhaps four) fourth-order coupling constants simply do not admit a stable fixed point. Were this conjecture to hold, the application of the fourth rule would be greatly simplified. Since L is directly calculable from the character table of the representation R, verification of this conjecture would represent an important step toward a rephrasing of the fourth rule directly in terms of R.

Unfortunately the conjecture is false. In this paper we construct LGW Hamiltonians possessing a stable fixed point for arbitrarily large L. The general Hamiltonian has an $n=2^p\cdot m$-component order parameter and $L=p+1$ invariants, where m and p are positive integers and $m>4$. We consider the cases $p=1$, $p=2$ and $p=3$, generalisation to higher p being obvious. For $p=1$ we introduce the $n=2m$-component order parameter

$$\boldsymbol{S}_i=(S_i^1, \ldots, S_i^m) \qquad i=1,2 \tag{2}$$

† The antiferromagnetic transitions in CeSe (Ott *et al* 1979) and CeTe appear experimentally to be second order, although the corresponding LGW model does not have a stable fixed point. Again it is possible that the transition is weakly first order (Mukamel and Wallace 1979).

‡ An nm-component vector model appropriate to cubic systems has been found to have a stable fixed point in the limit $n\to 0$ (Aharony 1975). This model has $L=4$ fourth-order invariants.

and construct the LGW Hamiltonian (1) with

$$f_0(\boldsymbol{S}_i) = \sum_{i=1}^{2} (\boldsymbol{S}_i \cdot \boldsymbol{S}_i)^2 \tag{3a}$$

$$f_1(\boldsymbol{S}_i) = (\boldsymbol{S}_1 \cdot \boldsymbol{S}_1)(\boldsymbol{S}_2 \cdot \boldsymbol{S}_2). \tag{3b}$$

It is clear that, starting with this Hamiltonian, one does not generate any new quartic invariants under RG transformation. This model has a *decoupled* fixed point at which (see e.g. Aharony 1976, Brezin *et al* 1976)

$$u_0^* = \frac{\varepsilon}{4K_4(m+8)} \qquad u_1^* = 0 \tag{4}$$

where K_4 is a phase space constant. When the initial value of u_1 is zero the fixed point (4) is, of course, stable. With respect to this fixed point the crossover exponent associated with u_1 is (Aharony 1976)

$$\lambda_1 = \alpha_m/\nu_m = (4-m)\varepsilon/(m+8) + \mathrm{O}(\varepsilon^2) \tag{5}$$

where α_m and ν_m are respectively the specific heat and correlation length critical exponents. For $m > 4$ one has $\lambda_1 < 0$; the *decoupled* fixed point is therefore stable even for non-zero initial values of u_1.

For $p = 2$ we affix an extra tensor index to $\boldsymbol{S}_i$, producing an $n = 4m$-component order parameter:

$$\boldsymbol{S}_{ij} = (S_{ij}^1, \ldots, S_{ij}^m) \qquad i, j = 1, 2.$$

The corresponding LGW model has $L = 3$ invariants defined as follows:

$$f_0(\boldsymbol{S}_{ij}) = \sum_{i,j=1}^{2} (\boldsymbol{S}_{ij} \cdot \boldsymbol{S}_{ij})^2 \tag{6a}$$

$$f_1(\boldsymbol{S}_{ij}) = \sum_{j=1}^{2} (\boldsymbol{S}_{1j} \cdot \boldsymbol{S}_{1j})(\boldsymbol{S}_{2j} \cdot \boldsymbol{S}_{2j}) \tag{6b}$$

$$f_2(\boldsymbol{S}_{ij}) = \sum_{i,i'=1}^{2} (\boldsymbol{S}_{i1} \cdot \boldsymbol{S}_{i1})(\boldsymbol{S}_{i'2} \cdot \boldsymbol{S}_{i'2}). \tag{6c}$$

Again it is trivial to verify that no new quartic invariants are generated by the RG transformation. With $u_2 = 0$ the model decomposes into decoupled $p = 1$ models, one involving $\boldsymbol{S}_{i1}$ and the other $\boldsymbol{S}_{i2}$. The fixed point (4) is then, as we have seen, stable for $m > 4$. The crossover exponent, λ_2, for u_2, with respect to this fixed point clearly satisfies

$$\lambda_2 = \lambda_1. \tag{7}$$

The fixed point (4) with $u_2^* = 0$ is therefore stable for $m > 4$, even when $u_2 \neq 0$ initially.

For $p = 3$ we affix an extra index to $\boldsymbol{S}_{ij}$, producing an $n = 8m$-component order parameter:

$$\boldsymbol{S}_{ijk} = (S_{ijk}^1, \ldots, S_{ijk}^m) \qquad i, j, k = 1, 2. \tag{8}$$

The corresponding LGW model has $L = 4$ invariants:

$$f_0(\boldsymbol{S}_{ijk}) = \sum_{i,j,k=1}^{2} (\boldsymbol{S}_{ijk} \cdot \boldsymbol{S}_{ijk})^2 \tag{9a}$$

$$f_1(\boldsymbol{S}_{ijk}) = \sum_{j,k=1}^{2} (\boldsymbol{S}_{1jk} \cdot \boldsymbol{S}_{1jk})(\boldsymbol{S}_{2jk} \cdot \boldsymbol{S}_{2jk}) \tag{9b}$$

$$f_2(\boldsymbol{S}_{ijk}) = \sum_{i,i',k=1}^{2} (\boldsymbol{S}_{i1k} \cdot \boldsymbol{S}_{i1k})(\boldsymbol{S}_{i'2k} \cdot \boldsymbol{S}_{i'2k}) \tag{9c}$$

$$f_3(\boldsymbol{S}_{ijk}) = \sum_{\substack{i,i' \\ j,j'=1}}^{2} (\boldsymbol{S}_{ij1} \cdot \boldsymbol{S}_{ij1})(\boldsymbol{S}_{i'j'2} \cdot \boldsymbol{S}_{i'j'2}). \tag{9d}$$

Once again verification that no new quartic invariants are generated under the RG is trivial. Again the decoupled fixed point (4) with $u_1^* = u_2^* = u_3^* = 0$ is stable for $m > 4$. It is trivial to extend this construction to arbitrary p; for each p the decoupled fixed point is stable and the number of invariants $L = p + 1$.

The symmetry properties of the general LGW Hamiltonian, H_p, with $(p+1)$ invariants, constructed in this way for any p, ensure that no new fourth-order invariants are generated by the RG. To see this, note that H_p is invariant under the group $G \equiv [O(m)]^{2^p} \times S$, where S is a subgroup of S_{2^p}, the permutation group of order 2^p. S is composed of the following elements of S_{2^p} and their products:

(i) all elements defined by the transformation

$$\boldsymbol{S}_{i_1,\ldots,i_{l-1},1,i_{l+1},\ldots,i_p} \rightarrow \boldsymbol{S}_{i'_1,\ldots,i'_{l-1},1,i_{l+1},\ldots,i_p} \qquad \boldsymbol{S}_{i_1,\ldots,i_{l-1},2,i_{l+1},\ldots,i_p} \rightarrow \boldsymbol{S}_{i_1,\ldots,i_{l-1},2,i_{l+1},\ldots,i_p} \tag{10}$$

where i'_j is either equal to i_j or to $\bar{i}_j$, with $\bar{i}_j$ defined as 2 if $i_j = 1$ and 1 if $i_j = 2$. Here l is an arbitrary integer satisfying $2 \leqslant l \leqslant p$;

(ii) all elements defined by interchanging 1 and 2 in the lth positions in equation (10);

(iii) all elements defined by the transformation

$$\boldsymbol{S}_{i_1,\ldots,i_p} \rightarrow \boldsymbol{S}_{i'_1,\ldots,i'_p}$$

where, again, for each j either $i'_j = i_j$ or $i'_j = \bar{i}_j$. It is simple to verify that for $m > 1$ the $(p+1)$ fourth-order terms of H_p specified by our construction are the only fourth-order invariants under the group G. Therefore no new invariants can be generated by the RG.

These models demonstrate the impossibility of formulating a sufficient condition for the nonexistence of a stable fixed point based solely on L. It is clearly of interest to find the symmetry criterion for the non-existence of a stable fixed point. This criterion evidently involves more than just the number of fourth-order invariants.

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